Oxford Cambridge and RSA

Monday 18 October 2021 – Afternoon

A Level Mathematics B (MEI)

H640/03 Pure Mathematics and Comprehension

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Time allowed: 2 hours

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INSTRUCTIONS
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INFORMATION
- This Insert contains the article for Section B.
- This document has **4** pages.

Adding arctangents

Where does the name 'arctangent' come from?

The two commonly used ways to denote the angle which has a tangent x are $\tan^{-1}x$ and $\arctan x$. The first of these is related to inverse function notation, $f^{-1}(x)$. Arctangent comes from radian measure, where an angle is represented by an arc on a unit circle; $\arctan x$ is the arc whose tangent is x.

An interesting result

It can be shown that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\arctan 1$.

Consider the diagram in **Fig. C1**.

Triangle ABC is right-angled at B.
AB = BC = 1 cm.
D is the midpoint of BC.

Using triangle ABD, $\tan\alpha=\frac{\text{DB}}{\text{BA}}=\frac{1}{2}$ so $\alpha=\arctan\left(\frac{1}{2}\right)$.

Using triangle ABC, $\tan(\alpha+\beta)=1$ so $\alpha+\beta=\arctan 1$.

Hence $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=1$.

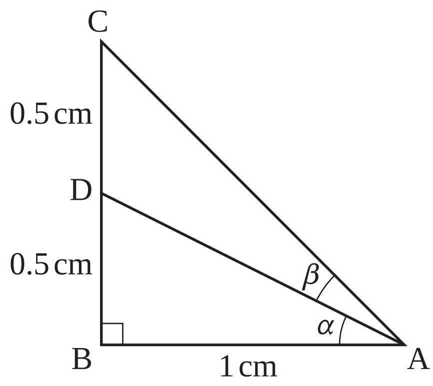

Fig. C1

Using $\tan\alpha=\frac{1}{2}$ and finding $\tan\beta$, it follows that $\beta=\arctan\left(\frac{1}{3}\right)$,

which gives the required result that $\arctan\left(\frac{1}{2}\right)+\arctan\left(\frac{1}{3}\right)=\arctan 1$.

Generalising the result

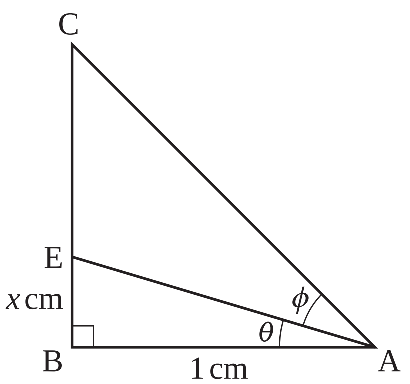

Fig. C2

Triangle ABC in **Fig. C2** is the same as triangle ABC in **Fig. C1** but E is a point on BC such that EB = x cm and $\theta=\arctan x$.

Following the same method as above, $\arctan x+\arctan\left(\frac{1-x}{1+x}\right)=\arctan 1$.

The arctan addition formula

The arctangent addition formula is a further generalization:

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right), \text{ as long as } xy < 1.$$

This result is equivalent to the addition formula $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$ where $\alpha = \arctan x$ and $\beta = \arctan y$.

To see why the restriction $xy < 1$ is necessary, consider what happens if $xy \geqslant 1$.

Clearly, $\dfrac{x+y}{1-xy}$ is undefined when $xy = 1$, so the formula does not apply in this case.

Suppose next that $xy > 1$, and that x and y are both positive; in this case $y > \dfrac{1}{x}$.

For any positive x, $\arctan x + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$.

$y > \dfrac{1}{x} \Rightarrow \arctan y > \arctan\left(\dfrac{1}{x}\right)$ so it follows that $\arctan x + \arctan y > \dfrac{\pi}{2}$.

However, $\arctan\left(\dfrac{x+y}{1-xy}\right)$ cannot be greater than $\dfrac{\pi}{2}$ as the range of the arctan function is $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

The formula $\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$ therefore cannot be valid in this case.

A similar argument can be used to show that the formula cannot be valid when $xy > 1$ and x and y are both negative.

If $xy > 1$, the arctangent addition formula needs to be adapted, as shown below.

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) - \pi, \text{ when } xy > 1 \text{ and } x, y < 0$$

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi, \text{ when } xy > 1 \text{ and } x, y > 0$$

Some additional results

- For n a positive integer, $\arctan\left(\dfrac{1}{n+1}\right) + \arctan\left(\dfrac{1}{n^2+n+1}\right) = \arctan\left(\dfrac{1}{n}\right)$; this follows directly from the arctan addition formula in line 23.
- $\arctan 1 + \arctan 2 + \arctan 3 = \pi$. This can be proved by using $\arctan x + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$ together with $\arctan\left(\dfrac{1}{2}\right) + \arctan\left(\dfrac{1}{3}\right) = \arctan 1$.